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On the exact region determined by Kendall's tau and Spearman's rho

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Abstract: Using properties of shuffles of copulas and tools from combinatorics we solve the open question about the exact region Ω determined by all possible values of Kendall's τ and Spearman's ρ. In particular, we prove that the well-known inequality established by Durbin and Stuart in 1951 is only sharp on a countable set with sole accumulation point $(-1,-1)$, give a simple analytic characterization of Ω in terms of a continuous, strictly increasing piecewise concave function, and show that Ω is compact and simply connected but not convex. The results also show that for each $(x,y) \in \Omega$ there are mutually completely dependent random variables whose τ and ρ values coincide with x and y respectively.

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1. Introduction

Kendall's τ and Spearman's ρ are, without doubt, the two most famous non-parametric measures of association/concordance. Given random variables X, Y with continuous distribution functions F and G respectively, Spearman's ρ is defined as the Pearson correlation coefficient of the $\mathcal{U}(0,1)$-distributed random variables $U := F \circ X$ and $V := G \circ Y$ whereas Kendall's τ is given by the probability of concordance minus the probability of discordance, i.e.

$$\begin{aligned}\rho(X,Y) &= 12\big(\mathbb{E}(UV) - \tfrac{1}{4}\big)\\ \tau(X,Y) &= \mathbb{P}\big((X_1-X_2)(Y_1-Y_2)>0\big) - \mathbb{P}\big((X_1-X_2)(Y_1-Y_2)<0\big),\end{aligned}$$

whereby (X_1,Y_1) and (X_2,Y_2) are independent and have the same distribution as (X,Y). Since both measures are scale invariant they only depend on the

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underlying (uniquely determined) copula A of (X, Y). It is well known and straightforward to verify (see [13]) that, given the copula A of (X, Y), Kendall's τ and Spearman's ρ can be expressed as

$$\tau(X, Y) = 4 \int_{[0,1]^2} A(x, y)\, d\mu_A(x, y) - 1 =: \tau(A) \tag{1.1}$$

$$\rho(X, Y) = 12 \int_{[0,1]^2} xy\, d\mu_A(x, y) - 3 =: \rho(A), \tag{1.2}$$

whereby μ_A denotes the doubly stochastic measure corresponding to A. Considering that τ and ρ quantify different aspects of the underlying dependence structure (see [6] and the references therein) a very natural question is how much they can differ, i.e. if $\tau(X, Y)$ is known which values may $\rho(X, Y)$ assume and vice versa. The following well-known universal inequalities between τ and ρ go back to Daniels [1] and Durbin and Stuart [4] respectively (for alternative proofs see [10, 7, 13]):

$$|3\tau - 2\rho| \leq 1 \tag{1.3}$$

$$\frac{(1+\tau)^2}{2} - 1 \leq \rho \leq 1 - \frac{(1-\tau)^2}{2} \tag{1.4}$$

The inequalities together yield the set Ω_0 (see Figure 1) to which we will refer to as *classical τ-ρ region* in the sequel. Daniels' inequality is known to be sharp (see [13]) whereas the first part of the inequality by Durbin and Stuart is only known to be sharp at the points $p_n = (-1+\frac{2}{n}, -1+\frac{2}{n^2})$ with $n \geq 2$ (which, using symmetry, is to say that the second part is sharp at the points $-p_n$). Although both inequalities are known since the 1950s and the interrelation between τ and ρ has received much attention also in recent years (see [6] and the references therein), to the best of the authors' knowledge the *exact τ-ρ region* Ω, defined by ($\mathcal{C}$ denoting the family of all two-dimensional copulas)

$$\begin{aligned} \Omega &= \big\{(\tau(X, Y), \rho(X, Y)) : \ X, Y \text{ continuous random variables}\big\} \\ &= \big\{(\tau(A), \rho(A)) : \ A \in \mathcal{C}\big\}, \end{aligned} \tag{1.5}$$

is still unknown.

In this paper we give a full characterization of Ω. We derive a piecewise concave, strictly increasing, continuous function $\Phi : [-1, 1] \to [-1, 1]$ and (see Theorem 3.5 and Theorem 5.1) prove that

$$\Omega = \big\{(x, y) \in [-1, 1]^2 : \Phi(x) \leq y \leq -\Phi(-x)\big\}. \tag{1.6}$$

Figure 1 depicts Ω_0 and the function Φ (lower red line), the explicit form of Φ is given in eq. (3.8) and eq. (3.7). As byproduct we get that the inequality by Durbin and Stuart is not sharp outside the aforementioned points p_n and $-pn$, that Ω is compact and simply connected, but not convex. Moreover, we prove the surprising fact that for each point $(x, y) \in \Omega$ there exist mutually completely dependent random variables X, Y for which $(\tau(X, Y), \rho(X, Y)) = (x, y)$ holds.

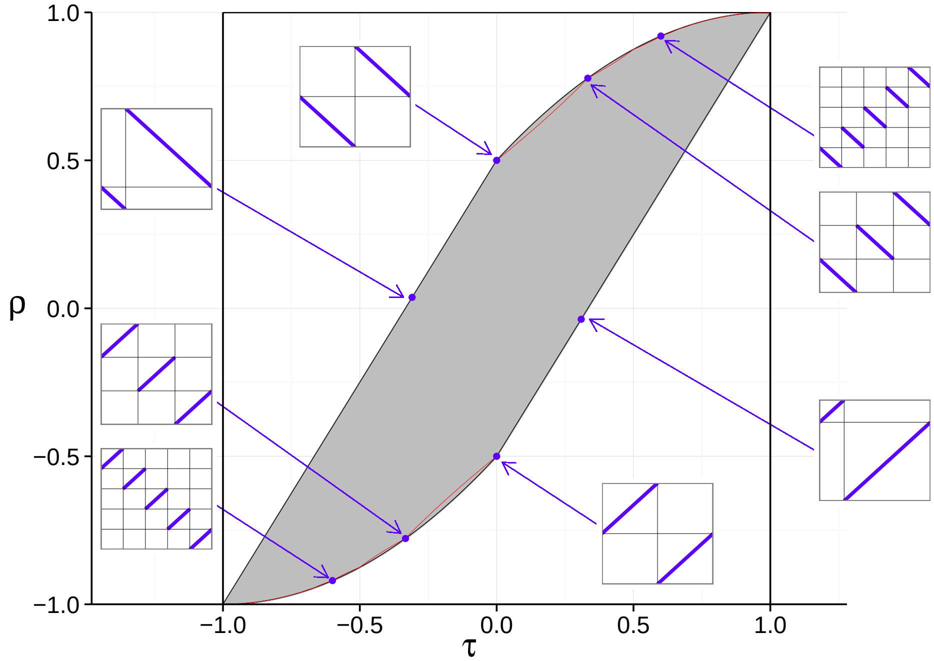

FIG 1. *The classical τ-ρ-region Ω_0 and some copulas (distributing mass uniformly on the blue segments) for which the inequality by Durbin and Stuart is sharp. The red line depicts the true boundary of Ω.*

The rest of the paper is organized as follows: Section 2 gathers some notations and preliminaries. In Section 3 we reduce the problem of determining Ω to a problem about so-called shuffles of copulas, prove some properties of shuffles and derive the function Φ. The main result saying that Ω is contained in the right-hand-side of eq. (1.6) is given in Section 4, tedious calculations needed for the proofs are collected to the Appendix. Finally, Section 5 serves to prove equality in eq. (1.6), and to collect some interesting consequences of this result.

2. Notation and Preliminaries

As already mentioned before, $\mathcal{C}$ will denote the family of all two-dimensional *copulas*, see [3, 5, 13, 16]. M and W will denote the minimum copula and the lower Fréchet-Hoeffding bound respectively. Given $A \in \mathcal{C}$ the *transpose* $A^t \in \mathcal{C}$ of A is defined by $A^t(x,y) := A(y,x)$ for all $x, y \in [0,1]$. d_∞ will denote the uniform distance on $\mathcal{C}$; it is well known that $(\mathcal{C}, d_\infty)$ is a compact metric space and that d_∞ is a metrization of weak convergence in $\mathcal{C}$. For every $A \in \mathcal{C}$ the corresponding *doubly stochastic measure* will be denoted by μ_A, i.e. we have $\mu_A([0,x] \times [0,y]) := A(x,y)$ for all $x, y \in [0,1]$. $\mathcal{P}_\mathcal{C}$ denotes the class of all

these doubly stochastic measures. $\mathcal{B}([0,1])$ and $\mathcal{B}([0,1]^2)$ will denote the Borel σ-fields in $[0,1]$ and $[0,1]^2$, λ and λ_2 the Lebesgue measure on $[0,1]$ and $[0,1]^2$ respectively. Instead of λ-a.e. we will simply write a.e. since no confusion will arise. $\mathcal{T}$ will denote the class of all λ-preserving transformations $h:[0,1]\to[0,1]$, i.e. transformations for which the push-forward λ^h of λ via h coincides with λ, $\mathcal{T}_b$ the subclass of all bijective $h\in\mathcal{T}$.

For every copula $A\in\mathcal{C}$ there exists a *Markov kernel* (regular conditional distribution) $K_A:[0,1]\times\mathcal{B}([0,1])\to[0,1]$ fulfilling ($G_x := \{y\in[0,1]:(x,y)\in G\}$ denoting the x-section of $G\in\mathcal{B}([0,1]^2)$ for every $x\in[0,1]$)

$$\int_{[0,1]} K_A(x,G_x)\,d\lambda(x)=\mu_A(G), \tag{2.1}$$

for every $G\in\mathcal{B}([0,1]^2)$, so, in particular

$$\int_{[0,1]} K_A(x,F)\,d\lambda(x)=\lambda(F) \tag{2.2}$$

for every $F\in\mathcal{B}([0,1])$. We will refer to K_A simply as Markov kernel of A. On the other hand, every Markov kernel $K:[0,1]\times\mathcal{B}([0,1])\to[0,1]$ fulfilling (2.2) induces a unique element $\mu\in\mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (2.1). For more details and properties of regular conditional distributions and disintegration we refer to [8, 9]. A copula $A\in\mathcal{C}$ will be called *completely dependent* if and only if there exists $h\in\mathcal{T}$ such that $K(x,E):=\mathbf{1}_E(h(x))$ is a Markov kernel of A (see [11, 19] for equivalent definitions and main properties). For every $h\in\mathcal{T}$ the induced completely dependent copula will be denoted by A_h. Note that $h_1=h_2$ a.e. implies $A_{h_1}=A_{h_2}$ and that eq. (2.1) implies $A_h(x,y)=\lambda([0,x]\cap h^{-1}([0,y]))$ for all $x,y\in[0,1]$. In the sequel $\mathcal{C}_d$ will denote the family of all completely dependent copulas. $A_h\in\mathcal{C}_d$ will be called *mutually completely dependent* if we even have $h\in\mathcal{T}_b$. Note that in case of $h\in\mathcal{T}_b$ we have $A_{h^{-1}}=(A_h)^t$. Complete dependence is the exact opposite of independence since it describes the (not necessarily mutual) situation of full predictability/maximum dependence. Some notions quantifying dependence of two-dimensional random variables which, contrary to Schweizer and Wolff's σ (see [15]) are not based on d_∞ have been studied in [18, 19].

Tackling the problem of determining the region Ω, our main tool will be special members of the class $\mathcal{C}_d$ usually referred to as *shuffles of* the minimum copula M. Following [13] we will call $h\in\mathcal{T}_b$ a *shuffle* (and $A_h\in\mathcal{C}_d$ a shuffle of M) if there exist $0=s_0<s_1<\ldots<s_{n-1}<s_n=1$ and $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n$ such that we have $h'(x)=\varepsilon_i$ for every $x\in(s_{i-1},s_i)$. In case of $\varepsilon_i=1$ for every $i\in\{1,\ldots,n\}$ we will call h *straight shuffle.* $\mathcal{S}$ will denote the family of all shuffles, $\mathcal{S}^+$ the family of all straight shuffles. It is well known (see [12, 13]) that $\mathcal{C}_{\mathcal{S}^+}$, defined by

$$\mathcal{C}_{\mathcal{S}^+}=\left\{A_h:\,h\in\mathcal{S}^+\right\} \tag{2.3}$$

is dense in $(\mathcal{C},d_\infty)$. For more general definitions of shuffles we refer to [3]. Obviously every shuffle $h\in\mathcal{S}$ can be expressed in terms of vectors $u\in\Delta_n,\varepsilon\in$

$\{-1,1\}^n$ and a permutation $\pi \in \sigma_n$, whereby Δ_n denotes the unit simplex $\Delta_n = \{x \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$ and σ_n denotes all bijections on $\{1,\ldots,n\}$. In fact, choosing suitable $u \in \Delta_n, \varepsilon \in \{-1,1\}^n, \pi \in \sigma_n$, setting (empty sums are zero by definition)

$$s_k := \sum_{i=1}^k u_i, \qquad t_k := \sum_{i=1}^k u_{\pi^{-1}(i)} \tag{2.4}$$

for every $k \in \{0,\ldots,n\}$, we have $s_k - s_{k-1} = u_k = t_{\pi(k)} - t_{\pi(k)-1}$ and on (s_{k-1}, s_k) the shuffle h is given by

$$h(x) = h_{\pi,u,\varepsilon}(x) := \begin{cases} t_{\pi(k)-1} + x - s_{k-1} & \text{if } \varepsilon_k = 1, \\ t_{\pi(k)} - (x - s_{k-1}) & \text{if } \varepsilon_k = -1. \end{cases} \tag{2.5}$$

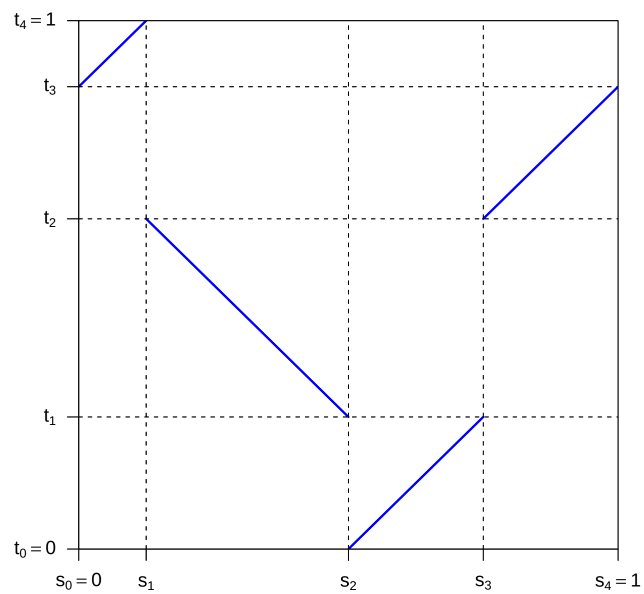

FIG 2. *Shuffle $h_{\pi,u,\varepsilon}$ with $\pi = (4,2,1,3), u = (\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4})$ and $\varepsilon = (1,-1,1,1)$*

In the sequel we will directly work with the function $h_{\pi,u,\varepsilon}$, implicitly defined in eq. (2.5) since all possible extensions of $h_{\pi,u,\varepsilon}$ from $\bigcup_{i=1}^k (s_{k-1}, s_k)$ to $[0,1]$ yield the same copula, which we will denote by $A_{h_{\pi,u,\varepsilon}}$. In case of $\varepsilon_i = 1$ for every $i \in \{1,\ldots,n\}$ we will simply write $h_{\pi,u}$ in the sequel. Note that the

chosen representation is not unique, i.e. for given $u \in \Delta_n, \varepsilon \in \{-1,1\}^n, \pi \in \sigma_n$ there always exist $u' \in \Delta_m, \varepsilon' \in \{-1,1\}^m, \pi' \in \sigma_m$ with $m \neq n$ such that $h_{\pi,u,\varepsilon} = h_{\pi',u',\varepsilon'}$ a.e., implying $A_{h_{\pi,u,\varepsilon}} = A_{h_{\pi',u',\varepsilon'}}$. So, for instance, the shuffle $h_{\pi,u,\varepsilon}$ with $\pi = (4,2,1,3), u = (\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4})$ and $\varepsilon = (1,-1,1,1)$ as depicted in Figure 2, and the shuffle $h_{\pi',u',\varepsilon'}$ with $\pi' = (5,3,1,2,4), u' = (\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4})$ and $\varepsilon = (1,-1,1,1,1)$ coincide a.e. and induce the same copula.

Remark 2.1. It might seem more natural to directly work with minimal representations (minimal dimension n) and to exclude the case of $u_k = 0$ for some k (implying $(s_{k-1}, s_k) = \emptyset$) in the first place – since we will, however, use various compactness arguments in the sequel the chosen representation is more convenient.

3. Basic properties of Ω and some results on shuffles

In this section we will first show that for determining Ω it is sufficient to consider straight shuffles, give explicit formulas for $(\tau(A_h), \rho(A_h))$ for arbitrary $h \in \mathcal{S}^+$, and derive a strictly increasing function $\Phi : [-1,1] \to [-1,1]$ which, after some change of coordinates, will finally be shown to fully determine Ω in the subsequent section.

We start with some observations about Ω. The mapping $f : \mathcal{C} \to [-1,1]^2$, defined by $f(A) = (\tau(A), \rho(A))$, is easily seen to be continuous w.r.t. d_∞, so compactness of $(\mathcal{C}, d_\infty)$ implies compactness of Ω. As a consequence, using eq. (1.5) and the fact that $\mathcal{C}_{\mathcal{S}^+}$ is dense we immediately get ($\overline{U}$ denoting the closure of a set U)

$$\Omega = \overline{\big\{(\tau(A_h), \rho(A_h)) : \, h \in \mathcal{S}^+\big\}}. \tag{3.1}$$

Based on this, our method of proof will be to construct a compact set Ω_Φ (fully determined by the function Φ) fulfilling $(\tau(A_h), \rho(A_h)) \in \Omega_\Phi$ for every $h \in \mathcal{S}^+$ since then we automatically get $\Omega \subseteq \Omega_\Phi$.

Apart from being compact, Ω is easily seen to be symmetric w.r.t. $(0,0)$. In fact, letting $\hat{A} \in \mathcal{C}$ denote the copula defined by $\hat{A}(x,y) = x - A(x, 1-y)$ for every $A \in \mathcal{C}$, both $\tau(\hat{A}) = -\tau(A)$ and $\rho(\hat{A}) = -\rho(A)$ follow immediately from eq. (1.1) and eq. (1.2). Having this, considering $\hat{\hat{A}} = A$, we obtain the stated symmetry w.r.t. $(0,0)$. Note that, setting $\hat{h} := 1 - h$, we get $\hat{A_h} = A_{\hat{h}}$ for every $h \in \mathcal{T}_b$. Analogously, it is straightforward to verify that $\tau(A^t) = \tau(A)$ as well as $\rho(A^t) = \rho(A)$ holds for every $A \in \mathcal{C}$, implying

$$\tau(A_{h^{-1}}) = \tau(A_h), \quad \rho(A_{h^{-1}}) = \rho(A_h) \tag{3.2}$$

for every $h \in \mathcal{T}_b$.

For every $h \in \mathcal{T}$ define the quantities $\mathrm{inv}(h)$ and $\mathrm{invsum}(h)$ (notation loosely based on [14]) by

$$\mathrm{inv}(h) = \int_{[0,1]^2} \mathbf{1}_{[0,x)}(y) \mathbf{1}_{(h(x),1]}(h(y)) \, d\lambda_2(x,y) \tag{3.3}$$

$$\mathrm{invsum}(h) = \int_{[0,1]^2} \mathbf{1}_{[0,x)}(y) \mathbf{1}_{(h(x),1]}(h(y)) (x-y) \, d\lambda_2(x,y). \tag{3.4}$$

Then the following result holds:

Lemma 3.1. *For every $h \in \mathcal{T}_b$ the following relations hold:*

$$
\begin{aligned}
\tau(A_h) &= 4\int_{[0,1]} A_h(x,h(x))\,d\lambda(x) - 1 = 1 - 4\,\mathrm{inv}(h)\\
\rho(A_h) &= 12\int_{[0,1]} xh(x)\,d\lambda(x) - 3 = 1 - 12\,\mathrm{invsum}(h)
\end{aligned}
$$

Moreover, for every $h \in \mathcal{T}_b$ we have $(\mathrm{inv}(h), \mathrm{invsum}(h)) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{6}\right]$.

Proof. Using disintegration we immediately get

$$
\begin{aligned}
\tau(A_h) &= 4\int_{[0,1]}\int_{[0,1]} A_h(x,y) K_{A_h}(x,dy)\,d\lambda(x) - 1\\
&= 4\int_{[0,1]} A_h(x,h(x))\,d\lambda(x) - 1
\end{aligned}
$$

as well as

$$
\begin{aligned}
\mathrm{inv}(h) &= \int_{[0,1]}\int_{[0,x]} \Big(1 - \mathbf{1}_{[0,h(x)]}(h(y))\Big)\,d\lambda(y)d\lambda(x)\\
&= \int_{[0,1]} \big(x - A_h(x,h(x))\big)\,d\lambda(x) = \frac{1-\tau(A_h)}{4}
\end{aligned}
$$

which proves the first identity. The first part of the second one is an immediate consequence of disintegration. To prove the remaining equality use $\int_{[0,x)} \mathbf{1}_{(h(x),1]}(h(y))\,d\lambda(y) = x - A_h(x,h(x))$ and $\int_{(y,1]} \mathbf{1}_{[0,h(y))}(h(x))\,d\lambda(x) = h(y) - A_h(y,h(y))$ to finally get

$$
\begin{aligned}
\mathrm{invsum}(A_h) &= \int_{[0,1]} x\big(x - A_h(x,h(x)) - (h(x) - A_h(x,h(x))\big)\,d\lambda(x)\\
&= \frac{1}{3} - \int_{[0,1]} xh(x)\,d\lambda(x).
\end{aligned}
$$

The fact that $(\mathrm{inv}(A_h), \mathrm{invsum}(A_h)) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{6}\right]$ is a direct consequence of $\Omega \subseteq [-1,1]^2$. ☐

As next step we derive explicit formulas for $\mathrm{inv}(h)$ and $\mathrm{invsum}(h)$ for the case of h being a straight shuffle based on which we will afterwards derive the aforementioned function Φ determining the region Ω. To simplify notation define

$$
\begin{aligned}
I_\pi &= \big\{\{i,j\} : 1 \le i < j \le n,\ \pi(i) > \pi(j)\big\}\\
Q_\pi &= \big\{\{i,j,k\} : 1 \le i < j < k \le n,\ \pi(i) > \pi(j) > \pi(k) \text{ or}\\
&\qquad \pi(j) > \pi(k) > \pi(i) \text{ or } \pi(k) > \pi(i) > \pi(j)\big\},
\end{aligned} \tag{3.5}
$$

as well as

$$
a_\pi(u) = \mathrm{inv}(h_{\pi,u}), \qquad b_\pi(u) = \mathrm{inv}(h_{\pi,u}) - 2\,\mathrm{invsum}(h_{\pi,u}) \tag{3.6}
$$

for every $\pi \in \sigma_n$ and $u \in \Delta_n$. The following lemma (the proof of which is given in the Appendix) holds.

Lemma 3.2. *For every $(\pi, u) \in \sigma_n \times \Delta_n$ the following identities hold:*

$$\begin{aligned}
\mathrm{inv}(h_{\pi,u}) &= a_\pi(u) = \sum_{i<j,\,\{i,j\}\in I_\pi} u_i u_j \\
\mathrm{invsum}(h_{\pi,u}) &= \sum_{i<j,\,\{i,j\}\in I_\pi} \left(\frac{1}{2}u_i^2 u_j + \frac{1}{2}u_i u_j^2 + \sum_{k:\, i<k<j} u_i u_j u_k \right) \\
b_\pi(u) &= \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} u_i u_j u_k
\end{aligned}$$

As pointed out in the Introduction, the first part of inequality (1.4) is known to be sharp only at the points $p_n = (-1 + \frac{2}{n}, -1 + \frac{2}{n^2})$ with $n \geq 2$. According to [13] (or directly using Lemma 3.2), considering $\pi = (n, n-1, \ldots, 2, 1)$ and $u_1 = u_2 = \ldots = u_n = \frac{1}{n}$ we get $p_n = (\tau(A_{h_{\pi,u}}), \rho(A_{h_{\pi,u}}))$. Having this, it seems natural to conjecture that all shuffles of the form $A_{h_{\pi,u}}$ with

$$\pi = (n, n-1, \ldots, 2, 1), \qquad u_1 = u_2 = \ldots = u_{n-1} = r,\ u_n = 1 - (n-1)r$$

for some $n \geq 2$ and $r \in (\frac{1}{n}, \frac{1}{n-1})$ might also be extremal in the sense that $(\tau(A_{h_{\pi,u}}), \rho(A_{h_{\pi,u}}))$ is a boundary point of Ω. Main content of the paper is the confirmation of this very conjecture. We will assign all shuffles of the just mentioned form the name prototype, calculate τ and ρ explicitly for all prototypes and then, based on these values, derive the function Φ.

Definition 3.3. $\pi \in \sigma_n$ *will be called* decreasing *if* $\pi = (n, n-1, \ldots, 2, 1)$*. The pair* $(\pi, u) \in \sigma_n \times \Delta_n$ *will be called a* prototype *if* π *is decreasing and there exists some* $r \in [\frac{1}{n}, \frac{1}{n-1}]$ *such that* $u_1 = u_2 = \ldots = u_{n-1} = r$ *and* $u_n = 1 - (n-1)r$*. Analogously,* $h \in \mathcal{S}^+$ *(and* $A_h \in \mathcal{C}_d$*) is called a* prototype *if there exists a prototype* (π, u) *such that* $h = h_{\pi,u}$ *a.e.*

Using the identities from Lemma 3.2 we get the following expressions for prototypes (the proof is given in the Appendix):

Lemma 3.4. *Suppose that $(\pi, u) \in \sigma_n \times \Delta_n$ is a prototype, then*

$$\begin{aligned}
\tau(A_{h_{\pi,u}}) &= 1 - 4(n-1)r + 2r^2 n(n-1) \in \left[\tfrac{2-n}{n}, \tfrac{2-(n-1)}{n-1}\right] \\
\rho(A_{h_{\pi,u}}) &= 1 - 2r(n-1)\big(3 - 3r(n-1) + r^2(n-2)n\big) \in \left[\tfrac{2-n^2}{n^2}, \tfrac{2-(n-1)^2}{(n-1)^2}\right].
\end{aligned}$$

Fix $n \geq 2$. Then both functions $r \mapsto 1 - 4(n-1)r + 2r^2 n(n-1)$ and $r \mapsto 1 - 2r(n-1)\big(3 - 3r(n-1) + r^2(n-2)n\big)$ are strictly increasing on $[\frac{1}{n}, \frac{1}{n-1}]$. Expressing r as function of τ and substituting the result in the expression for ρ directly yields

$$\rho(A_{h_{\pi,u}}) = -1 - \frac{4}{n^2} + \frac{3}{n} + \frac{3\tau(A_{h_{\pi,u}})}{n} - \frac{n-2}{\sqrt{2}n^2\sqrt{n-1}}(n - 2 + n\tau(A_{h_{\pi,u}}))^{3/2}.$$

Based on this interrelation define $\Phi_n : [-1+\frac{2}{n}, 1] \to [-1,1]$ by

$$\Phi_n(x) = -1 - \frac{4}{n^2} + \frac{3}{n} + \frac{3x}{n} - \frac{n-2}{\sqrt{2}n^2\sqrt{n-1}}(n-2+nx)^{3/2} \tag{3.7}$$

and set

$$\Phi(x) = \begin{cases} -1 & \text{if } x = -1, \\ \Phi_n(x) & \text{if } x \in \left[\frac{2-n}{n}, \frac{2-(n-1)}{n-1}\right] \text{ for some } n \geq 2. \end{cases} \tag{3.8}$$

Since we have $\Phi_n(\frac{2-n}{n}) = \Phi_{n+1}(\frac{2-n}{n}) = -1 + \frac{2}{n^2}$ for every $n \geq 1$ this defines a function $\Phi : [-1,1] \to [-1,1]$. Notice that $\Phi_2(x) = -\frac{1}{2} + \frac{3x}{2}$, i.e. on $[0,1]$ Φ coincides with Daniels' linear bound and for $x_n = \frac{2-n}{n}$ and $n \geq 2$ we have $(x_n, \Phi(x_n)) = p_n$, i.e. $(x_n, \Phi(x_n))$ coincides with the points at which Durbin and Stuart's inequality is known to be sharp. Furthermore, it is straightforward to verify that Φ is a strictly increasing homemorphism on $[-1,1]$ which is concave on every interval $[\frac{2-n}{n}, \frac{2-(n-1)}{n-1}]$ with $n \geq 2$. Figure 3 depicts the function Φ as well as some prototypes and their corresponding Kendall's τ and Spearman's ρ. Defining the compact set Ω_Φ by

$$\Omega_\Phi = \big\{(x,y) \in [-1,1]^2 : \Phi(x) \leq y \leq -\Phi(-x)\big\}, \tag{3.9}$$

we can now state the following main result the proof of which is given in the next section.

Theorem 3.5. *The precise τ-ρ region Ω fulfills $\Omega \subseteq \Omega_\Phi$.*

Remark 3.6. The fact that $\Omega \subseteq \Omega_\Phi$ holds is the principal result of this paper since it improves the classical inequality by Durbin and Stuart mentioned in the Introduction and, more importantly, gives sharp bounds everywhere. In Section 5 we will, however, show that even $\Omega = \Omega_\Phi$ holds and that for every point $(x,y) \in \Omega$ there exists a shuffle $h \in \mathcal{S}$ such that $(\tau(A_h), \rho(A_h)) = (x,y)$.

Remark 3.7. A function similar (but not identical) to Φ has appeared in the literature in [17], where the authors tried to deduce sharp bounds of Ω by running simulations (but did not provide any analytic proof). Additionally, it has been brought to our attention during the preparation of this manuscript that Manuel Úbeda-Flores (University of Almería) already conjectured Theorem 3.5 (with the exact form of Φ) in a working paper in 2009.

4. Proof of the main theorem

Using the properties of Ω mentioned at the beginning of Section 3, Theorem 3.5 is proved if we can show that for every $h \in \mathcal{S}^+$ we have $\rho(A_h) \geq \Phi(\tau(A_h))$. Given Lemma 3.2 it is straightforward to verify that this is equivalent to showing $\mathrm{invsum}(h) \leq \varphi(\mathrm{inv}(h))$ for every $h \in \mathcal{S}^+$ whereby $\varphi : [0, \frac{1}{2}] \to [0, \frac{1}{6}]$ is defined by

$$\varphi(x) = \begin{cases} \frac{1}{6} & \text{if } x = \frac{1}{2}, \\ \varphi_n(x) & \text{if } x \in [\frac{1}{2} - \frac{1}{2(n-1)}, \frac{1}{2} - \frac{1}{2n}] \text{ for some } n \geq 2 \end{cases} \tag{4.1}$$

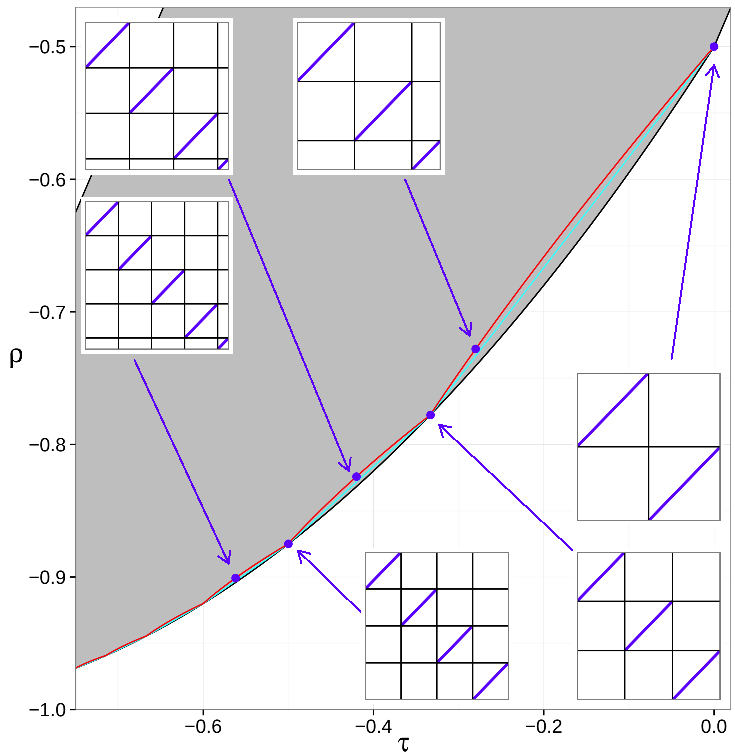

FIG 3. *The function Φ (red) and some prototypes with their corresponding Kendall's τ and Spearman's ρ. The shaded region depicts the classical τ-ρ-region Ω_0, straight lines connecting the points p_n are plotted in green.*

and $\varphi_n : [\frac{1}{2} - \frac{1}{2(n-1)}, \frac{1}{2} - \frac{1}{2n}] \to [0, \frac{1}{6}]$ is given by

$$\varphi_n(x) = \frac{1}{6} + \frac{1}{3n^2} - \frac{1}{2n} + \frac{x}{n} + \frac{n-2}{6n^2\sqrt{n-1}}(n-1-2nx)^{3/2}. \tag{4.2}$$

Translating this to $a_\pi(u)$ and $b_\pi(u)$, using eq. (3.6) and defining $\vartheta : [0, \frac{1}{2}] \to [0, \frac{1}{6}]$ by $\vartheta(x) = x - 2\varphi(x)$ we arrive at the following equivalent form of Theorem 3.5:

Theorem 4.1. *For every $n \in \mathbb{N}$, $\pi \in \sigma_n$ and $u \in \Delta_n$ the following inequality holds:*

$$b_\pi(u) \geq \vartheta(a_\pi(u)) \tag{4.3}$$

We are now going to prove this result and start with some first observations and an outline of the structure of the subsequent proof. (i) ϑ is continuous and, by calculating the derivative, it is straightforward to see that ϑ is non-decreasing. (ii) $\vartheta(0) = \vartheta(\frac{1}{4}) = 0$ and $\vartheta(\frac{1}{2}) = \frac{1}{6}$. (iii) For every prototype (π, u) we have equality $b_\pi(u) = \vartheta(a_\pi(u))$. (iv) For given n and fixed $\pi \in \sigma_n$ the functions $v \mapsto a_\pi(v)$ and $v \mapsto b_\pi(v)$ are continuous on Δ_n, so there exists some

$u \in \Delta_n$ minimizing the function $v \mapsto b_\pi(v) - \vartheta(a_\pi(v))$. (v) For $n \leq 2$ the inequality $b_\pi(u) \geq \vartheta(a_\pi(u))$ trivially holds for every $\pi \in \sigma_n$ and every $u \in \Delta_n$, so **from now on we will only consider the case** $n \geq 3$.

The structure of the proof of Theorem 4.1 is as follows:

1. Preliminary Step 1: We prove inequality (4.3) for the case of decreasing $\pi \in \sigma_n$.
2. Preliminary Step 2: We analyze how, for fixed $\pi \in \sigma_n$, the quantities $a_\pi(u)$ and $b_\pi(u)$ change if $u \in \Delta_n$ changes.
3. Induction Step 1: Assuming that the result is true for all $(\pi, u) \in \sigma_m \times \Delta_m$ with $m < n$ we prove inequality (4.3) for $(\pi, u) \in \sigma_n \times \Delta_n$ under the hypothesis that there either exist (i) $p < q < r$ such that $\pi(r) > \pi(q) > \pi(p)$ or (ii) $p < q < r < s$ such that $\pi(q) > \pi(p) > \pi(s) > \pi(r)$ holds.
4. Induction Step 2: Assuming that the result is true for all $(\pi, u) \in \sigma_m \times \Delta_m$ with $m < n$ we prove inequality (4.3) for $(\pi, u) \in \sigma_n \times \Delta_n$ with π not fulfilling the hypothesis in Induction Step I.

Preliminary Step 1: Consider $n \geq 3$ and $\pi = (n, n-1, \ldots, 2, 1)$. Note that in this situation we have $e_1(u) = 1, e_2(u) = a_\pi(u), e_3(u) = b_\pi(u)$ for every $u \in \Delta_n$, whereby e_i denotes the i-th elementary symmetric polynomial for $i \in \{1, 2, 3\}$, i.e. $e_1(v) := \sum_i v_i$, $e_2(v) := \sum_{i<j} v_i v_j$ and $e_3(v) := \sum_{i<j<k} v_i v_j v_k$ for every $v \in \mathbb{R}^n$. Hence $a_\pi(u)$ and $b_\pi(u)$ do not change if we reorder the coordinates of u.

Lemma 4.2. *Suppose that $n \geq 3, \pi = (n, n-1, \ldots, 2, 1), c_2 \in a_\pi(\Delta_n)$ and that $u \in \Delta_n$ fulfills $b_\pi(u) = \min\{b_\pi(v) : v \in \Delta_n \cap (a_\pi)^{-1}(\{c_2\})\}$ as well as $u_1 \geq \cdots \geq u_n \geq 0$. Then there exists $m \in \{1, \ldots, n\}$ such that $u_i = 0$ for every $i > m$, and $u_1 = \cdots = u_{m-1} \geq u_m$.*

Proof. Note that continuity of b_π and compactness of $\Delta_n \cap (a_\pi)^{-1}(\{c_2\})$ implies the existence of the minimum. We first prove the statement for the case $n = 3$ and suppose that u is a minimizer fulfilling $u_3 \geq u_2 \geq u_1 \geq 0$. Define a polynomial $f : \mathbb{R} \to \mathbb{R}$ by

$$f(T) = (T - u_1)(T - u_2)(T - u_3) = T^3 - T^2 + c_2 T - e_3(u),$$

and let D_f denote the discriminant of f. It is well known that $D_f > 0$ if and only if f has three distinct real zeros and that in case of $D_f \neq 0$ locally the zeros of f are smooth (so in particular continuous) functions of the coefficients of f.

Suppose that $u_1 > u_2 > u_3 > 0$. Then $D_f > 0$. Let $f_\varepsilon(T) = T^3 - T^2 + c_2 T - (e_3(u) - \epsilon)$, then for small enough values of $\epsilon > 0$, the polynomial f_ϵ has three distinct, positive real zeros: $u_{\epsilon,1}, u_{\epsilon,2}, u_{\epsilon,3}$. Then $u_{\epsilon,1} + u_{\epsilon,2} + u_{\epsilon,3} = 1$ and $u_{\epsilon,1}u_{\epsilon,2} + u_{\epsilon,2}u_{\epsilon,3} + u_{\epsilon,3}u_{\epsilon,1} = c_2$, while $u_{\epsilon,1}u_{\epsilon,2}u_{\epsilon,3} = e_3(u) - \varepsilon < e_3(u)$, contradiction. So either $u_3 = 0$ or $u_1 = u_2$ or $u_1 > u_2 = u_3 > 0$. In the first two cases we are done, so suppose that $u_1 > u_2 = u_3 > 0$. Then $1 = u_1 + 2u_2$ and $c_2 = 2u_1u_2 + u_2^2$. Suppose that $u_1 \geq 4u_2$. Then $1 \geq 4c_2$, so there are

unique $y_1 \geq y_2 \geq 0$ such that $y_1 + y_2 = 1$ and $y_1 y_2 = c_2$. Let $y_3 = 0$, then considering $y = (y_1, y_2, y_3)$ we get $e_1(y) = 1$, $e_2(y) = c_2$, and $e_3(y) = 0 < e_3(u)$, contradiction. So $4u_2 > u_1 > u_2$. Let $y_1 = y_2 = \frac{2u_1+u_2}{3}$ and $y_3 = \frac{4u_2-u_1}{3}$. Then $y_1, y_2, y_3 \geq 0$, $e_1(y) = 1$, $e_2(y) = c_2$, and $e_3(y) = \frac{1}{27}(2u_1 + u_2)^2(4u_2 - u_1) = e_3(u) - \frac{4}{27}(u_1 - u_2)^3 < e_3(u)$, contradiction. This proves the claim for $n = 3$.

Suppose indirectly that the statement is false for some $n > 3$. Then there are $i < j < k$ such that $u_i > u_j \geq u_k > 0$. Setting $\bar{u}_l := \frac{u_l}{u_i+u_j+u_k}$ for every $l \in \{i, j, k\}$ obviously $\bar{u}_i + \bar{u}_j + \bar{u}_k = 1$. Applying the case $n = 3$ to $\bar{u}_i, \bar{u}_j, \bar{u}_k$ yields $\bar{y}_i, \bar{y}_j, \bar{y}_k \in [0, 1]$ such that $\bar{y}_i + \bar{y}_j + \bar{y}_k = \bar{u}_i + \bar{u}_j + \bar{u}_k$, $\bar{y}_i\bar{y}_j + \bar{y}_j\bar{y}_k + \bar{y}_k\bar{y}_i = \bar{u}_i\bar{u}_j + \bar{u}_j\bar{u}_k + \bar{u}_k\bar{u}_i$ and $\bar{y}_i\bar{y}_j\bar{y}_k < \bar{u}_i\bar{u}_j\bar{u}_k$. Setting $y_l = u_l$ for every $l \in \{1, \ldots, n\} \setminus \{i, j, k\}$ and $y_l = \bar{y}_l(u_i + u_j + u_k)$ for every $l \in \{i, j, k\}$ finally yields $e_1(y) = e_1(u)$, $e_2(y) = e_2(u)$ and $e_3(y) < e_3(u)$, contradiction. □

Corollary 4.3. *Suppose that $n \geq 3$ and that $\pi = (n, n-1, \ldots, 2, 1)$. Then $b_\pi(u) \geq \vartheta(a_\pi(u))$ holds for every $u \in \Delta_n$.*

Preliminary Step 2: We investigate how, for fixed $\pi \in \sigma_n$, the quantities $a_\pi(u)$ and $b_\pi(u)$ change if $u \in \Delta_n$ changes. To do so, temporarily extend a_π and b_π to full $\mathbb{R}^n$ using the identities in Lemma 3.2. The following lemmata (whose proof is given in the Appendix) will be crucial in the sequel.

Lemma 4.4. *Suppose that $n \geq 3$ and that $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ fulfills $\sum_i \delta_i = 0$. Then for every $t \in \mathbb{R}$ the following identities hold:*

$$a_\pi(u + t\delta) - a_\pi(u) = \alpha_1 t + \alpha_2 t^2 \tag{4.4}$$

$$b_\pi(u + t\delta) - b_\pi(u) = \beta_1 t + \beta_2 t^2 + \beta_3 t^3 \tag{4.5}$$

where

$$\alpha_1 = \sum_i a_i \delta_i, \qquad \alpha_2 = \sum_{i<j,\ \{i,j\} \in I_\pi} \delta_i \delta_j,$$

$$\beta_1 = \sum_i b_i \delta_i, \qquad \beta_2 = \sum_{i<j} c_{i,j} \delta_i \delta_j, \qquad \beta_3 = \sum_{i<j<k, \{i,j,k\} \in Q_\pi} \delta_i \delta_j \delta_k,$$

and

$$a_i = \sum_{j\colon \{i,j\} \in I_\pi} u_j, \qquad b_i = \sum_{j<k,\ \{i,j,k\} \in Q_\pi} u_j u_k, \qquad c_{i,j} = c_{j,i} = \sum_{k\colon \{i,j,k\} \in Q_\pi} u_k.$$

Lemma 4.5. *Suppose that $n \geq 3$ and that $\pi \in \sigma_n$. If $p, q, r \in \{1, \ldots, n\}$ are distinct elements such that $\{p, q, r\} \notin Q_\pi$, then $c_{p,r} + c_{q,r} \geq c_{p,q} \geq 0$.*

We now state two conditions for π that imply the existence of a direction $\delta \in \mathbb{R}^n \setminus \{0\}$ with $\sum_i \delta_i = 0$ such that $t \mapsto a_\pi(u + t\delta) - a_\pi(u)$ is identical to zero for every t and $t \mapsto b_\pi(u + t\delta) - b_\pi(u)$ is of degree two and concave.

Lemma 4.6. *Suppose that $n \geq 3$, that $\pi \in \sigma_n$, and that one of the following two conditions holds:*

(i) There exist $p, q, r \in \{1, 2, \ldots, n\}$ with $p < q < r$ and $\pi(r) > \pi(q) > \pi(p)$.

(ii) *There exist* $p, q, r, s \in \{1, 2, \dots, n\}$ *with* $p < q < r < s$ *and* $\pi(q) > \pi(p) > \pi(s) > \pi(r)$.

Then there exists $\delta \in \mathbb{R}^n \setminus \{0\}$ *such that the coefficients in (4.4) and (4.5) fulfill* $\alpha_1 = \alpha_2 = \beta_3 = 0$ *and* $\beta_2 \le 0$.

Induction Step 1: We prove the induction step for all $\pi \in \sigma_n$ fulfilling one of the conditions in Lemma 4.6.

Lemma 4.7. *Suppose that* $n \ge 3$ *and that* $b_\omega(v) \ge \vartheta(a_\omega(v))$ *holds for all* $(\omega, v) \in \sigma_m \times \Delta_m$ *with* $m < n$. *If* $\pi \in \sigma_n$ *fulfills one of the conditions in Lemma 4.6 then* $b_\pi(u) \ge \vartheta(a_\pi(u))$ *for every* $u \in \Delta_n$.

Proof. Suppose that $\pi \in \sigma_n$ fulfills one of the conditions in Lemma 4.6 and consider $u \in \Delta_n$. If $u_k = 0$ for some $k \in \{1, \dots, n\}$ then, defining $(\pi', v) \in \sigma_{n-1} \times \Delta_{n-1}$ by $v_i = u_i$ for $i < k$ and $v_i = u_{i+1}$ for $i \ge k$ as well as

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i < k \text{ and } \pi(i) < \pi(k) \\ \pi(i) - 1 & \text{if } i < k \text{ and } \pi(i) > \pi(k), \\ \pi(i+1) & \text{if } i \ge k \text{ and } \pi(i+1) < \pi(k) \\ \pi(i+1) - 1 & \text{if } i \ge k \text{ and } \pi(i+1) > \pi(k), \end{cases}$$

we immediately get $b_\pi(u) = b_{\pi'}(v) \ge \vartheta(a_{\pi'}(v)) = \vartheta(a_\pi(u))$.

Suppose now that $u \in (0,1)^n$ and, using Lemma 4.6, choose $\delta \in \mathbb{R}^n \setminus \{0\}$ such that $\beta_2 \le 0$ and $a_\pi(u + t\delta) = a_\pi(u)$ and $b_\pi(u + t\delta) - b_\pi(u) = \beta_1 t + \beta_2 t^2$ for all $t \in \mathbb{R}$. Considering $u \in (0,1)^n$ there are $t_0 < 0 < t_1$ such that $u + t\delta \in [0,1]^n$ if and only if $t \in [t_0, t_1]$. Concavity of $t \mapsto b_\pi(u + t\delta)$ implies that $b_\pi(u + t_0\delta) \le b_\pi(u)$ or $b_\pi(u + t_1\delta) \le b_\pi(u)$. Moreover there are i, j such that $(u + t_0\delta)_i = 0$ and $(u + t_1\delta)_j = 0$ by construction, so we can proceed as in the first step of the proof and use induction to get $b_\pi(u) \ge \vartheta(a_\pi(u))$. ☐

Induction Step 2: As final step we concentrate on permutations $\pi \in \sigma_n$ not fulfilling any of the two conditions in 4.6 and start with the following definition and the subsequent lemma (whose proof can be found in the Appendix).

Definition 4.8. *A permutation* $\pi \in \sigma_l$ *is called* almost decreasing *if there is at most one* $i \in \{1, \dots, l-1\}$ *such that* $\pi(i) < \pi(i+1)$.

Lemma 4.9. *Let* $l \ge 1$ *and* $\pi \in \sigma_l$. *Then the following two conditions are equivalent:*

- *There are no* $1 \le p < q < r \le l$ *such that* $\pi(p) < \pi(q) < \pi(r)$, *and there are no* $1 \le p < q < r < s \le l$ *such that* $\pi(r) < \pi(s) < \pi(p) < \pi(q)$.
- π *or* π^{-1} *is almost decreasing.*

Having this characterization we can now prove the remaining induction step for those $\pi \in \sigma_n$ fulfilling that π or π^{-1} is almost decreasing. Notice that w.l.o.g. we may assume that $\pi \in \sigma_n$ is almost decreasing since defining $v \in \Delta_n$ by $v_i = u_{\pi^{-1}(i)}$ for every $i \in \{1, \dots, n\}$ yields $a_\pi(u) = a_{\pi^{-1}}(v)$ as well as $b_\pi(u) = b_{\pi^{-1}}(v)$. Both subsequent lemmata are therefore only stated and proved for almost decreasing π.

Lemma 4.10. *Suppose that $n \geq 3$ and that $b_\omega(v) \geq \vartheta(a_\omega(v))$ holds for all $(\omega, v) \in \sigma_m \times \Delta_m$ with $m < n$. If $\pi \in \sigma_n$ is almost decreasing with $\pi(1) = n$ or $\pi(n) = 1$ then $b_\pi(u) \geq \vartheta(a_\pi(u))$ holds for every $u \in \Delta_n$.*

Proof. As before we may assume $u \in (0,1)^n$. Suppose that $\pi(1) = n$. Defining $(\pi', u') \in \sigma_{n-1} \times \Delta_{n-1}$ by $\pi'(i) = \pi(i+1)$ and $u'_i = \frac{u_{i+1}}{1-u_1}$ for every $i \in \{1, \ldots, n-1\}$ and considering

$$a_{\pi'}(u') = \frac{1}{(1-u_1)^2} \sum_{2 \leq i < j \leq n \colon \{i,j\} \in I_\pi} u_i u_j$$

yields that $a_\pi(u) = (1-u_1)^2 a_{\pi'}(u') + u_1(1-u_1)$. Analogously, using

$$b_{\pi'}(u') = \frac{1}{(1-u_1)^3} \sum_{2 \leq i < j < k \leq n \colon \{i,j,k\} \in Q_\pi} u_i u_j u_k$$

we get $b_\pi(u) = (1-u_1)^3 b_{\pi'}(u') + u_1(1-u_1)^2 a_{\pi'}(u')$. To simplify notation let $\tilde{\pi}_k$ denote the decreasing permuation in σ_k for every $k \in \mathbb{N}$. Choose $u''_1, ..., u''_{n-1} \in \Delta_{n-1}$ such that $a_{\tilde{\pi}_{n-1}}(u'') = a_{\pi'}(u')$ and $b_{\tilde{\pi}_{n-1}}(u'') = \vartheta(a_{\pi'}(u'))$. Define $\tilde{u} = (\tilde{u}_1, ..., \tilde{u}_n)$ by $\tilde{u}_1 = u_1$ and $\tilde{u}_i = (1-u_1)u''_{i-1}$ for every $i \in \{2, \ldots, n\}$. Then $\sum_{i=1}^n \tilde{u}_i = u_1 + (1-u_1)\sum_{i=1}^{n-1} u''_i = 1$ and we get

$$a_{\tilde{\pi}_n}(\tilde{u}) = (1-u_1)^2 a_{\tilde{\pi}_{n-1}}(u'') + u_1(1-u_1) = a_\pi(u)$$

as well as

$$\begin{aligned} b_{\tilde{\pi}_n}(\tilde{u}) &= \sum_{1<i<j<k \colon \{i,j,k\} \in Q_{\tilde{\pi}_n}} \tilde{u}_i \tilde{u}_j \tilde{u}_k + \tilde{u}_1 \sum_{1<j<k \colon \{j,k\} \in I_{\tilde{\pi}_n}} \tilde{u}_i \tilde{u}_j \\ &= (1-u_1)^3 b_{\tilde{\pi}_{n-1}}(u'') + u_1(1-u_1)^2 a_{\tilde{\pi}_{n-1}}(u''). \end{aligned}$$

Altogether this yields

$$\begin{aligned} b_\pi(u) &= (1-u_1)^3 b_{\pi'}(u') + u_1(1-u_1)^2 a_{\pi'}(u') \\ &\geq (1-u_1)^3 \vartheta(a_{\pi'}(u')) + u_1(1-u_1)^2 a_{\pi'}(u') \\ &= (1-u_1)^3 b_{\tilde{\pi}_{n-1}}(u'') + u_1(1-u_1)^2 a_{\tilde{\pi}_{n-1}}(u'') = b_{\tilde{\pi}_n}(\tilde{u}) \\ &\geq \vartheta(a_{\tilde{\pi}_n}(\tilde{u})) = \vartheta(a_\pi(u)). \end{aligned}$$

The proof of the case $\pi(n) = 1$ is completely analogous. $\square$

The following final lemma assures that in case of almost decreasing $\pi \in \sigma_n$ with $\pi(1) \neq n$ and $\pi(n) \neq 1$ we can not be on the boundary of Ω_Φ. Note that in the proof we do not make use of the induction hypothesis.

Lemma 4.11. *Suppose that $n \geq 3$ and that $\pi \in \sigma_n$ is almost decreasing with $\pi(1) \neq n$ and $\pi(n) \neq 1$. Then for every $u \in \Delta_n \cap (0,1)^n$ we have*

$$b_\pi(u) - \vartheta(a_\pi(u)) > \min\Big\{ b_\omega(v) - \vartheta(a_\omega(v)) : \omega \in \sigma_n,\, v \in \Delta_n \Big\}$$

Proof. First note that the existence of the minimum is assured by the fact that σ_n is finite and Δ_n is compact. Set $k := \pi^{-1}(1)$. Then $1 = \pi(k) < \pi(k-1) < \cdots < \pi(1) < n$ and $1 < \pi(n) < \cdots < \pi(k+2) < \pi(k+1)$, so $\pi(k+1) = n$. Define $(\pi', u') \in \sigma_n \times \Delta_n$ as follows: $\pi'(i) = \pi(i)$ for $i \notin \{k, k+1\}$, $\pi'(k) = \pi(k+1) = n$ and $\pi'(k+1) = \pi(k) = 1$; $u'_i = u_i$ for $i \notin \{k, k+1\}$, $u'_k = u_{k+1}$ and $u'_{k+1} = u_k$. Then it is straightforward to verify that

$$a_{\pi'}(u') - a_\pi(u) = u_k u_{k+1}$$

and

$$b_{\pi'}(u') - b_\pi(u) = -\sum_{i \neq k, k+1} u_k u_{k+1} u_i,$$

holds, which, considering $n \geq 3$ implies $a_{\pi'}(u') > a_\pi(u)$ and $b_{\pi'}(u') < b_\pi(u)$. Having this we get $b_\pi(u) - \vartheta(a_\pi(u)) > b_{\pi'}(u') - \vartheta(a_{\pi'}(u'))$ since ϑ is non-decreasing, which completes the proof. □

Since Lemma 4.11 implies that in order to prove inequality (4.3) for every $\pi \in \sigma_n$ and $u \in \Delta_n$ it is not necessary to consider almost decreasing permutations π with $\pi(1) \neq n$ and $\pi(n) \neq 1$ the proof of Theorem 4.1 (hence the one of Theorem 3.5) is complete.

5. Additional related results

So far we have shown that $\Omega \subseteq \Omega_\Phi$. We now prove that the two sets are in fact identical.

Theorem 5.1. *The precise τ-ρ region Ω coincides with Ω_Φ. In particular, Ω is not convex.*

Proof. The construction of Φ implies the existence of a family $(A_t)_{t \in [0,1]}$ of shuffles of M fulfilling that the map $t \mapsto A_t$ is continuous on $[0,1]$ (w.r.t. d_∞) and that

$$\gamma(t) := (\tau(A_t), \rho(A_t)) = \begin{cases} (4t-1, \Phi(4t-1)) & \text{if } t \in [0, \frac{1}{2}] \\ (3-4t, -\Phi(4t-3)) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{5.1}$$

Obviously the curve $\gamma : [0,1] \to [-1,1]^2$ is simply closed and rectifiable. For every $s \in [0,1]$ consider similarities $f_s, g_s : [0,1]^2 \to [0,1]^2$, given by $f_s(x,y) = s(x,y)$ and $g_s(x,y) = (1-s)(x,y) + (s,s)$ and define the (ordinal sum) operator $O_s : \mathcal{C} \to \mathcal{C}$ implicitly via

$$\mu_{O_s(A)} = s\mu_M^{f_s} + (1-s)\mu_A^{g_s}.$$

Then we have $d_\infty(O_s(A), O_s(B)) \leq d_\infty(A,B)$ for all $A, B \in \mathcal{C}$ and every $s \in [0,1]$, and the mapping $s \mapsto O_s(A)$ is continuous for every $A \in \mathcal{C}$. Consequently, the function $H : [0,1]^2 \to [-1,1]^2$, given by

$$H(s,t) = \big(\tau(O_s(A_t)), \rho(O_s(A_t))\big)$$

is continuous and fulfills, firstly, that $H(0,t) = \gamma(t)$ and $H(1,t) = (1,1)$ for every $t \in [0,1]$ and, secondly, that $H(s,0) = H(s,1)$ for all $s \in [0,1]$. In other words, H is a homotopy and γ is homotopic to the constant curve $(1,1)$, implying $\Omega = \Omega_\Phi$. Since Φ is strictly concave on each interval $[\frac{2-n}{n}, \frac{2-(n-1)}{n-1}]$ with $n \geq 3$, $\Omega = \Omega_\Phi$ cannot be convex. $\square$

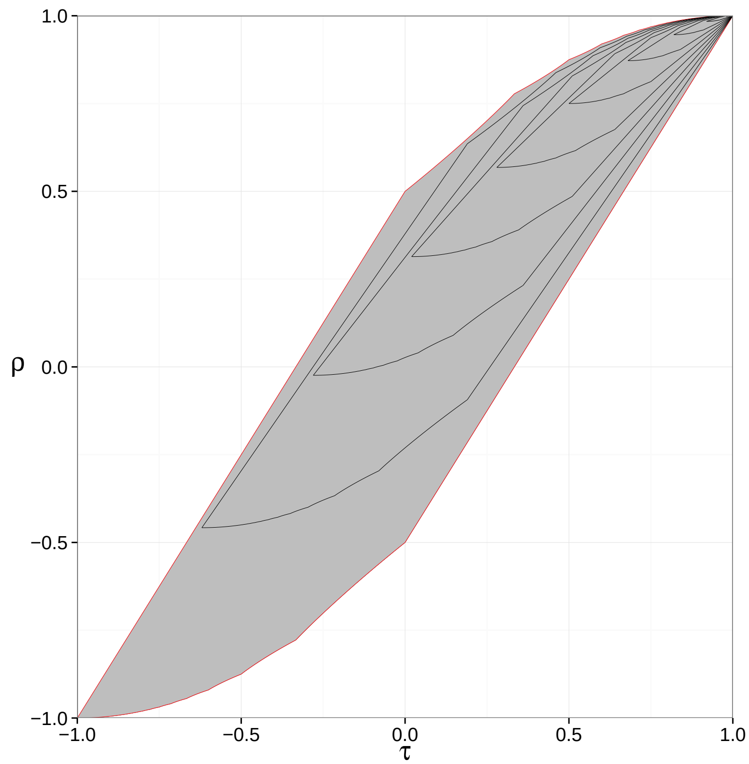

FIG 4. *The curves $\gamma_s(t) = H(s,t)$ for $t \in [0,1]$ and $s \in \{\frac{1}{10}, \dots, \frac{9}{10}\}$ with H being the homotopy used in the proof of Theorem 5.1.*

Considering that the operator $O_s : \mathcal{C} \to \mathcal{C}$ maps the family of all shuffles of M into itself for every $s \in [0,1]$ the proof of Theorem 5.1 has the following surprising byproduct:

Corollary 5.2. *For every point $(x,y) \in \Omega$ there is a shuffle $h \in \mathcal{S}$ such that $(\tau(A_h), \rho(A_h)) = (x,y)$.*

Additionally, Theorem 5.1 also implies the following result concerning the possible range of Spearman's ρ if Kendall's τ is known (and vice versa):

Corollary 5.3. *Suppose that X, Y are continuous random variables with $\tau(X,Y) = \tau_0$. Then $\rho(X,Y) \in [\Phi(\tau_0), -\Phi(-\tau_0)]$.*

Remark 5.4. Due to the simple analytic form of Φ is straightforward to verify that

$$\lambda_2(\Omega) = \frac{4}{5} - \frac{4}{5}\zeta(3) + \frac{2}{15}\pi^2 \approx 1.1543,$$

whereby $\zeta(3) = \sum_{i=1}^{\infty} \frac{1}{i^3}$. Considering that $\lambda_2(\Omega_0) = \frac{7}{6} \approx 1.1667$ this underlines the quality of the classical inequalities.

6. Appendix

Proof of Lemma 3.2. Since the first identity is straightforward to verify we start with the proof of the second one. Using $s_j^2 - s_{j-1}^2 = (s_j - s_{j-1})(s_j + s_{j-1}) = u_j(2\sum_{k<j} u_k + u_j)$ we get

$$\begin{aligned}
\mathrm{invsum}(h_{\pi,u}) &= \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \int_0^1 \mathbf{1}_{[0,y)}(x)\mathbf{1}_{(h_{\pi,u}(y),1]}(h_{\pi,u}(x))(y-x)\,d\lambda(x)\,d\lambda(y) \\
&= \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \sum_{\substack{i:\, i<j,\\ \pi(i)>\pi(j)}} \int_{s_{i-1}}^{s_i} (y-x)\,d\lambda(x)\,d\lambda(y) \\
&= \sum_{j=1}^{n} \sum_{\substack{i:\, i<j,\\ \pi(i)>\pi(j)}} \int_{s_{j-1}}^{s_j} \left(y\,u_i - \frac{1}{2}u_i^2 - u_i \sum_{k:\, k<i} u_k \right) d\lambda(y) \\
&= \sum_{j=1}^{n} \sum_{\substack{i:\, i<j,\\ \pi(i)>\pi(j)}} \left(\frac{1}{2}u_i u_j \Big(2\sum_{l:\, l<j} u_l + u_j \Big) - \frac{1}{2}u_i^2 u_j - u_i u_j \sum_{k:\, k<i} u_k \right) \\
&= \sum_{\substack{i<j:\\ \pi(i)>\pi(j)}} \left(u_i u_j \sum_{l:\, l<j} u_l + \frac{1}{2}u_i u_j^2 - \frac{1}{2}u_i^2 u_j - u_i u_j \sum_{k:\, k<i} u_k \right) \\
&= \sum_{\substack{i<j:\\ \pi(i)>\pi(j)}} \left(\frac{1}{2}u_i u_j^2 + \frac{1}{2}u_i^2 u_j + \sum_{k:\, i<k<j} u_i u_j u_k \right).
\end{aligned}$$

The third identity follows from

$$\begin{aligned}
b_\pi(u) &= \mathrm{inv}(h_{\pi,u}) - 2\,\mathrm{invsum}(h_{\pi,u}) \\
&= \sum_{k<i<j,\, \pi(i)>\pi(j)} u_i u_j u_k + \sum_{i<k<j,\, \pi(i)>\pi(j)} u_i u_j u_k + \sum_{i<j<k,\, \pi(i)>\pi(j)} u_i u_j u_k \\
&\quad + \sum_{i<j,\, \pi(i)>\pi(j)} u_i^2 u_j + \sum_{i<j,\, \pi(i)>\pi(j)} u_i u_j^2 \\
&\quad - \sum_{i<j,\, \pi(i)>\pi(j)} u_i^2 u_j - \sum_{i<j,\, \pi(i)>\pi(j)} u_i u_j^2 - 2 \sum_{i<k<j,\, \pi(i)>\pi(j)} u_i u_j u_k
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i<j<k,\, \pi(j)>\pi(k)} u_i u_j u_k + \sum_{i<j<k,\, \pi(i)>\pi(j)} u_i u_j u_k - \sum_{i<j<k,\, \pi(i)>\pi(k)} u_i u_j u_k \\
&= \sum_{\substack{i<j<k,\\ \pi(i)>\pi(j)>\pi(k)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(j)>\pi(i)>\pi(k)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(j)>\pi(k)>\pi(i)}} u_i u_j u_k \\
&\quad + \sum_{\substack{i<j<k,\\ \pi(k)>\pi(i)>\pi(j)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(i)>\pi(k)>\pi(j)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(i)>\pi(j)>\pi(k)}} u_i u_j u_k \\
&\quad - \sum_{\substack{i<j<k,\\ \pi(j)>\pi(i)>\pi(k)}} u_i u_j u_k - \sum_{\substack{i<j<k,\\ \pi(i)>\pi(j)>\pi(k)}} u_i u_j u_k - \sum_{\substack{i<j<k,\\ \pi(i)>\pi(k)>\pi(j)}} u_i u_j u_k \\
&= \sum_{\substack{i<j<k,\\ \pi(i)>\pi(j)>\pi(k)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(j)>\pi(k)>\pi(i)}} u_i u_j u_k + \sum_{\substack{i<j<k,\\ \pi(k)>\pi(i)>\pi(j)}} u_i u_j u_k.
\end{aligned}
$$

□

Proof of Lemma 3.4. To simplify calculations let $e_i(v)$ denote the i-th elementary symmetric polynomial for $i \in \{1,2,3\}$ and $v \in \mathbb{R}^n$, i.e. $e_1(v) = \sum_i v_i$, $e_2(v) = \sum_{i<j} v_i v_j$ and $e_3(v) = \sum_{i<j<k} v_i v_j v_k$. Using $\sum_i v_i^2 = e_1(v)^2 - 2e_2(v)$ it follows that

$$
\begin{aligned}
\operatorname{inv}(h_{\pi,u}) = e_2(u) &= \frac{1}{2}\Big(e_1(u)^2 - \sum_i u_i^2\Big) = \frac{1}{2}\Big(1 - \sum_i u_i^2\Big) \\
&= \frac{1}{2}\Big(1 - (n-1)r^2 - (1-(n-1)r)^2\Big) = r(n-1) - \frac{1}{2}r^2 n(n-1).
\end{aligned}
$$

Moreover, considering $r \in [\frac{1}{n}, \frac{1}{n-1}]$ we get $\operatorname{inv}(h_{\pi,u}) \in [\frac{1}{2} - \frac{1}{2(n-1)}, \frac{1}{2} - \frac{1}{2n}]$, implying $\tau(A_{h_{\pi,u}}) \in [-1 + \frac{2}{n}, -1 + \frac{2}{n-1}]$, which completes the proof of the first assertion. Using $\sum_i v_i^3 = e_1(v)^3 - 3e_1(v)e_2(v) + 3e_3(v)$ and Lemma 3.2 moreover it follows that

$$
\begin{aligned}
\operatorname{invsum}(h_{\pi,u}) &= \frac{1}{2}\operatorname{inv}(h_{\pi,u}) - \frac{1}{2}\, b_\pi(h) = \frac{1}{2}e_2(u) - \frac{1}{2}e_3(u) \\
&= \frac{1}{2}e_2(u) - \frac{1}{2}\Big(\frac{1}{3}\sum_i u_i^3 - \frac{1}{3}e_1(u)^3 + e_1(u)e_2(u)\Big) \\
&= \frac{1}{2}e_2(u) - \frac{1}{6}\sum_i u_i^3 + \frac{1}{6} - \frac{1}{2}e_2(u) = \frac{1}{6} - \frac{1}{6}\sum_i u_i^3 \\
&= \frac{1}{6} - \frac{1}{6}\big((n-1)r^3 + (1-(n-1)r)^3\big).
\end{aligned}
$$

Again considering $r \in [\frac{1}{n}, \frac{1}{n-1}]$ we get $\operatorname{invsum}(h_{\pi,u}) \in [\frac{1}{6} - \frac{1}{6(n-1)^2}, \frac{1}{6} - \frac{1}{6n^2}]$, implying $\rho(A_{h_{\pi,u}}) \in [-1 + \frac{2}{n^2}, -1 + \frac{2}{(n-1)^2}]$, which completes the proof. □

Proof of Lemma 4.4. The expression for $a_\pi(u+t\delta)-a_\pi(u)$ is easily verified:

$$\begin{aligned}
a_\pi(u+t\delta)-a_\pi(u) &= \sum_{i<j,\,\{i,j\}\in I_\pi} ((u_i+\delta_i t)(u_j+\delta_j t)-u_iu_j) \\
&= t\sum_{i<j,\,\{i,j\}\in I_\pi} \delta_j u_i+\delta_i u_j + t^2 \sum_{i<j,\,\{i,j\}\in I_\pi} \delta_i\delta_j \\
&= t\left(\sum_{j<i,\,\{i,j\}\in I_\pi} \delta_i u_j + \sum_{i<j,\,\{i,j\}\in I_\pi} \delta_i u_j\right) + t^2\alpha_2 \\
&= t\sum_{i=1}^n \delta_i \sum_{j:\,\{i,j\}\in I_\pi} u_j + t^2\alpha_2 \\
&= t\sum_{i=1}^n \delta_i a_i + t^2\alpha_2 = \alpha_1 t+\alpha_2 t^2
\end{aligned}$$

To derive the expression for $b_\pi(u+t\delta)-b_\pi(u)$ notice that

$$\begin{aligned}
b_\pi(u+t\delta)-b_\pi(u) &= \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} ((u_i+\delta_i t)(u_j+\delta_j t)(u_k+\delta_k t)-u_iu_ju_k) \\
&= \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} \delta_i\delta_j\delta_k t^3+\delta_i\delta_j u_k t^2+\delta_i\delta_k u_j t^2 \\
&\quad +\delta_j\delta_k u_i t^2+\delta_k u_iu_j t+\delta_j u_iu_k t+\delta_i u_ju_k t \\
&= t^3 \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} \delta_i\delta_j\delta_k \\
&\quad + t^2 \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} \delta_i\delta_j u_k+\delta_i\delta_k u_j+\delta_j\delta_k u_i \\
&\quad + t \sum_{i<j<k,\,\{i,j,k\}\in Q_\pi} \delta_k u_iu_j+\delta_j u_iu_k+\delta_i u_ju_k \\
&= \beta_1 t+\beta_2 t^2+\beta_3 t^3
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_i\delta_j u_k+\delta_i\delta_k u_j+\delta_j\delta_k u_i \\
&\qquad = \sum_{\substack{i<j<k:\\ \{i,k,j\}\in Q_\pi}} \delta_i\delta_j u_k + \sum_{\substack{i<k<j:\\ \{i,k,j\}\in Q_\pi}} \delta_i\delta_j u_k + \sum_{\substack{k<i<j:\\ \{k,i,j\}\in Q_\pi}} \delta_i\delta_j u_k \\
&\qquad = \sum_{i<j}\delta_i\delta_j \left(\sum_{\substack{i<j<k:\\ \{i,k,j\}\in Q_\pi}} u_k + \sum_{\substack{i<k<j:\\ \{i,k,j\}\in Q_\pi}} u_k + \sum_{\substack{k<i<j:\\ \{k,i,j\}\in Q_\pi}} u_k\right) \\
&\qquad = \sum_{i<j}\delta_i\delta_j \sum_{k:\,\{i,k,j\}\in Q_\pi} u_k
\end{aligned}$$

$$= \sum_{i<j} \delta_i \delta_j c_{i,j} = \beta_2$$

and

$$\begin{aligned}
&\sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_k u_i u_j + \delta_j u_i u_k + \delta_i u_j u_k \\
&\qquad = \sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_k u_i u_j + \sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_j u_i u_k + \sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_i u_j u_k \\
&\qquad = \sum_{\substack{j<k<i:\\ \{i,j,k\}\in Q_\pi}} \delta_i u_j u_k + \sum_{\substack{j<i<k:\\ \{i,j,k\}\in Q_\pi}} \delta_i u_j u_k + \sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} \delta_i u_j u_k \\
&\qquad = \sum_{i=1}^n \delta_i \left(\sum_{\substack{j<k<i:\\ \{i,j,k\}\in Q_\pi}} u_j u_k + \sum_{\substack{j<i<k:\\ \{i,j,k\}\in Q_\pi}} u_j u_k + \sum_{\substack{i<j<k:\\ \{i,j,k\}\in Q_\pi}} u_j u_k \right) \\
&\qquad = \sum_{i=1}^n \delta_i \sum_{j<k} u_j u_k = \sum_{i=1}^n \delta_i b_i = \beta_1.
\end{aligned}$$

$\square$

Proof of Lemma 4.5. The inequality $c_{p,q} \geq 0$ immediately follows from $u_1, \dots, u_n \geq 0$. Let

$$\iota(i,j) = \iota(j,i) = \begin{cases} 1 & \text{if } (i<j \text{ and } \pi(i)>\pi(j)) \text{ or } (j<i \text{ and } \pi(j)>\pi(i)), \\ -1 & \text{otherwise} \end{cases}$$

and

$$\gamma_{i,j,k} = \frac{1}{2}(1 + \iota(i,j)\iota(i,k)\iota(j,k)) \in \{0,1\}$$

for every $i,j,k \in \{1,\dots,n\}$. Then we have $\{i,j,k\} \in Q_\pi$ if and only if $\iota(i,j)\iota(j,k)\iota(i,k) = 1$ if and only if $\gamma_{i,j,k} = 1$. Therefore $c_{i,j} = \sum_k \gamma_{i,j,k} u_k$ for every i,j. So

$$c_{p,r} + c_{q,r} - c_{p,q} = \sum_i (\gamma_{p,r,i} + \gamma_{q,r,i} - \gamma_{p,q,i}) u_i$$

and it is enough to prove $\gamma_{p,r,i} + \gamma_{q,r,i} \geq \gamma_{p,q,i}$ for every i. If $\gamma_{p,q,i} = 0$ or $\gamma_{p,r,i} = 1$ or $\gamma_{q,r,i} = 1$, then this is clear. So suppose indirectly that $\gamma_{p,q,i} = 1$ and $\gamma_{p,r,i} = \gamma_{q,r,i} = 0$. Then $\iota(p,q)\iota(p,i)\iota(q,i) = 1$, $\iota(p,r)\iota(p,i)\iota(r,i) = -1$ and $\iota(q,r)\iota(q,i)\iota(r,i) = -1$. Multiplying these together, we get that $\iota(p,q)\iota(p,r)\iota(q,r) = 1$, so $\gamma_{p,q,r} = 1$. However $\{p,q,r\} \notin Q_\pi$, so $\gamma_{p,q,r} = 0$, contradiction. $\square$

Proof of Lemma 4.6. (i) Suppose that there are $p < q < r$ such that $\pi(r) > \pi(q) > \pi(p)$. Let $\delta_i = 0$ for every $i \neq p,q,r$. We can fix a nonzero solution $(\delta_p, \delta_q, \delta_r) \in \mathbb{R}^3 \setminus \{0\}$ to the following system of homogeneous linear equations:

$$\delta_p + \delta_q + \delta_r = 0, \qquad a_p\delta_p + a_q\delta_q + a_r\delta_r = 0.$$

Then $\sum_i \delta_i = 0$, $\alpha_1 = 0$, moreover $\{p,q\},\{p,r\},\{q,r\} \notin I_\pi$ and $\{p,q,r\} \notin Q_\pi$, so $\alpha_2 = \beta_3 = 0$. The numbers $\delta_p\delta_q$, $\delta_p\delta_r$, $\delta_q\delta_r$ cannot be all negative, so e.g., $\delta_p\delta_q \geq 0$. Then using Lemma 4.5 we obtain

$$\begin{aligned}\beta_2 &= c_{p,q}\delta_p\delta_q + c_{p,r}\delta_p\delta_r + c_{q,r}\delta_q\delta_r\\ &\leq (c_{p,r}+c_{q,r})\delta_p\delta_q + c_{p,r}\delta_p\delta_r + c_{q,r}\delta_q\delta_r\\ &= c_{p,r}\delta_p(\delta_q+\delta_r) + c_{q,r}\delta_q(\delta_p+\delta_r) = -c_{p,r}\delta_p^2 - c_{q,r}\delta_q^2 \leq 0.\end{aligned}$$

Now suppose that there are $p<q<r<s$ such that $\pi(q) > \pi(p) > \pi(s) > \pi(r)$. Let $\delta_i = 0$ for $i \neq p,q,r,s$. We can fix a nonzero solution $(\delta_p,\delta_q,\delta_r,\delta_s) \in \mathbb{R}^4 \setminus \{0\}$ to the following system of homogeneous linear equations:

$$\delta_p + \delta_q = 0, \qquad \delta_r + \delta_s = 0, \qquad a_p\delta_p + a_q\delta_q + a_r\delta_r + a_s\delta_s = 0.$$

Then $\sum_i \delta_i = 0$, $\alpha_1 = 0$, and

$$\alpha_2 = \delta_p\delta_r + \delta_p\delta_s + \delta_q\delta_r + \delta_q\delta_s = (\delta_p+\delta_q)(\delta_r+\delta_s) = 0.$$

Moreover $\{p,q,r\},\{p,q,s\},\{p,r,s\},\{q,r,s\} \notin Q_\pi$, so $\beta_3 = 0$. We claim that

$$\beta_2 = -c_{p,q}\delta_p^2 + (c_{p,r}+c_{q,s}-c_{p,s}-c_{q,r})\delta_p\delta_r - c_{r,s}\delta_r^2 \leq 0.$$

Let $d = c_{p,r}+c_{q,s}-c_{p,s}-c_{q,r}$. Lemma 4.5 implies that $c_{p,q} \geq |c_{q,r}-c_{p,r}|$ and $c_{p,q} \geq |c_{p,s}-c_{q,s}|$, so $2c_{p,q} \geq |c_{q,r}-c_{p,r}| + |c_{p,s}-c_{q,s}| \geq |d|$. Similarly, $c_{r,s} \geq |c_{p,r}-c_{p,s}|$ and $c_{r,s} \geq |c_{q,s}-c_{q,r}|$, so $2c_{r,s} \geq |d|$ too. Since either $-d \leq 0$ or $d \leq 0$, the equations

$$\begin{aligned}2\beta_2 &= -2c_{p,q}\delta_p^2 + 2d\delta_p\delta_r - 2c_{r,s}\delta_r^2\\ &= -d(\delta_p-\delta_r)^2 - (2c_{p,q}-d)\delta_p^2 - (2c_{r,s}-d)\delta_r^2 =\\ &= d(\delta_p+\delta_r)^2 - (2c_{p,q}+d)\delta_p^2 - (2c_{r,s}+d)\delta_r^2\end{aligned}$$

imply that $\beta_2 \leq 0$. □

Proof of Lemma 4.9. Each of the two conditions is true for π if and only if it is true for π^{-1}. It is easy to see that the second condition implies the first one. Conversely, suppose that π (and hence also π^{-1}) satisfies the first condition. We prove by induction on l. The statement is trivial for $l = 1$, so let $l \geq 2$. If $\pi(l) = 1$ then we can use the induction hypothesis for $\pi|_{\{1,\dots,l-1\}} \in \sigma_{l-1}$. If $\pi(1) = l$ then we can use the induction hypothesis for $\pi' \in \sigma_{l-1}$, where $\pi'(i) = \pi(i+1) - 1$ for every $i \in \{1,\dots,l-1\}$. So we may assume $\pi(1) \neq l$ and $\pi(l) \neq 1$.

Suppose that $\pi^{-1}(l) > \pi^{-1}(1)$ and set $k = \pi^{-1}(l)$. If $\pi^{-1}(1) < k-1 < k$, then $1 = \pi(\pi^{-1}(1)) < \pi(k-1) < \pi(k) = l$, which contradicts the condition on π. Consider $\pi^{-1}(1) = k-1$. If $i<j<k$, then we cannot have $\pi(i) < \pi(j)$, because then we would have $\pi(i) < \pi(j) < \pi(k) = l$, contradicting the condition on π. If $k-1<i<j$ then we cannot have $\pi(i) < \pi(j)$, because then we would have $1 = \pi(\pi^{-1}(1)) = \pi(k-1) < \pi(i) < \pi(j)$, contradicting the condition on π. So $\pi(i) > \pi(i+1)$ for every $i \in \{1,\dots,l-1\} \setminus \{k\}$, hence π is almost decreasing.

Now suppose that $\pi^{-1}(l) < \pi^{-1}(1)$. If $\pi(l) < \pi(1)$, then the condition on π is false for $p = 1$, $q = \pi^{-1}(l)$, $r = \pi^{-1}(1)$, $s = l$. So $\pi(l) > \pi(1)$ and $(\pi^{-1})^{-1}(l) > (\pi^{-1})^{-1}(1)$. Applying the previous paragraph to π^{-1} shows that π^{-1} is almost decreasing. □

References

[1] Daniels, H. E. (1950). Rank correlation and population models. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **12** 171–191. MR0040629

[2] Durante, F. and Sarkoci, P. and Sempi, C. (2009). Shuffles of copulas. *J. Math. Anal. Appl.* **352** 914–921. MR2501937

[3] Durante, F. and Sempi, C. (2010). Copula theory: an introduction. In *Copula theory and its applications, Lecture Notes in Statistics–Proceedings* (P. Jaworski, F. Durante, W. Härdle, T. Rychlik) 1–314. Springer, Berlin.

[4] Durbin, J. and Stuart, A. (1951). Inversions and Rank Correlation Coefficients *J. R. Stat. Soc. Ser. B Stat. Methodol.* **13** 303–309. MR0047994

[5] Embrechts, P. and Lindskog, F. and McNeil, A. (2001). Modelling Dependence with Copulas and Applications to Risk Management. In *Handbook of Heavy Tailed Distributions in Finance* (S. T. Rachev) 329–384. Elsevier, Amsterdam.

[6] Fredricks, G. A. and Nelsen, R. B. (2007). On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables. *J. Statist. Plann. Inference* **137** 2143–2150. MR2325421

[7] Genest, C. and Nešlehová, J. (2009). Analytical proofs of classical inequalities between Spearman's ρ and Kendall's τ. *J. Statist. Plann. Inference* **139** 3795-3798. MR2553765

[8] Kallenberg, O. (1997). *Foundations of modern probability.* Springer, New York Berlin Heidelberg. MR1464694

[9] Klenke, A. (2008). *Probability Theory - A Comprehensive Course.* Springer, Berlin Heidelberg. MR2372119

[10] Kruskal, W. H. (1958). Ordinal measures of association. *J. Amer. Statist. Assoc.* **53** 814–861. MR0100941

[11] Lancaster, H. O. (1963). Correlation and complete dependence of random variables. *Ann. Statist.* **34** 1315–1321. MR0154376

[12] Mikusínski, P. and Sherwood, H. and Taylor, M. D. (1992). Shuffles of Min. *Stochastica* **13** 61–74. MR1197328

[13] Nelsen, R. B. (2006). *An Introduction to Copulas.* Springer, New York. MR2197664

[14] Sack, J. and Úlfarsson, H. (2012). Refined inversion statistics on permutations. *Electron. J. Combin.* **19** P29. MR2880660

[15] Schweizer, E. and Wolff, E.F. (1981). On nonparametric measures of dependence for random variables. *Ann. Stat.* **9** 879-885. MR0619291

[16] Sempi, C. (2011). Copulae: Some mathematical aspects. *Appl. Stoch. Models Bus. Ind.* **27** 37–50. MR2752452

[17] Shao, W. and Guo, G. and Zhao, G. and Meng, F. (2014). Simulated

annealing for the bounds of Kendall's τ and Spearman's ρ. *J. Stat. Comput. Simul.* **84** 2688–2699. MR3250967

[18] Siburg, K. F. and Stoimenov, P. A. (2010). A measure of mutual complete dependence. *Metrika* **71** 239–251. MR2602190

[19] Trutschnig, W. (2011). On a strong metric on the space of copulas and its induced dependence measure. *J. Math. Anal. Appl.* **384** 690–705. MR2825218